THE COLLATZ CONJECTURE

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Mathematics is not yet ripe enough for such questions

Paul Erdos - 1983

Outline of presentation

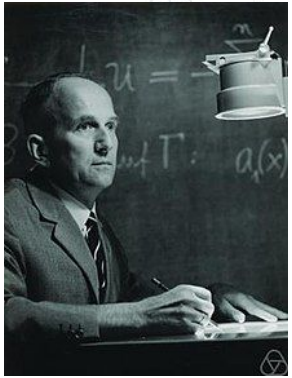
1. Definition
   - ➔ Define Collatz sequences
   - ➔ Define Collatz conjecture
2. Significant results
   - ➔ Mean Downward trend
   - ➔ Modification of the conjecture
   - ➔ Benford’s Law
   - ➔ Collatz circles
3. Collatz fractal and Tao Observation
   - ➔ Collatz Fractal
   - ➔ Tao Observation

Definition

Introduced by Lothar Collatz ➡ 1937

- Define the function *Col* on the positive integers {1,2,3,...} by the following rules.

$$Col(n) = \begin{cases} 3n+1 \text{ if n is odd} \\ n/2 \text{ if n is even} \end{cases}$$



Lothar Collatz
1910-1990
(Photo courtesy MFO)

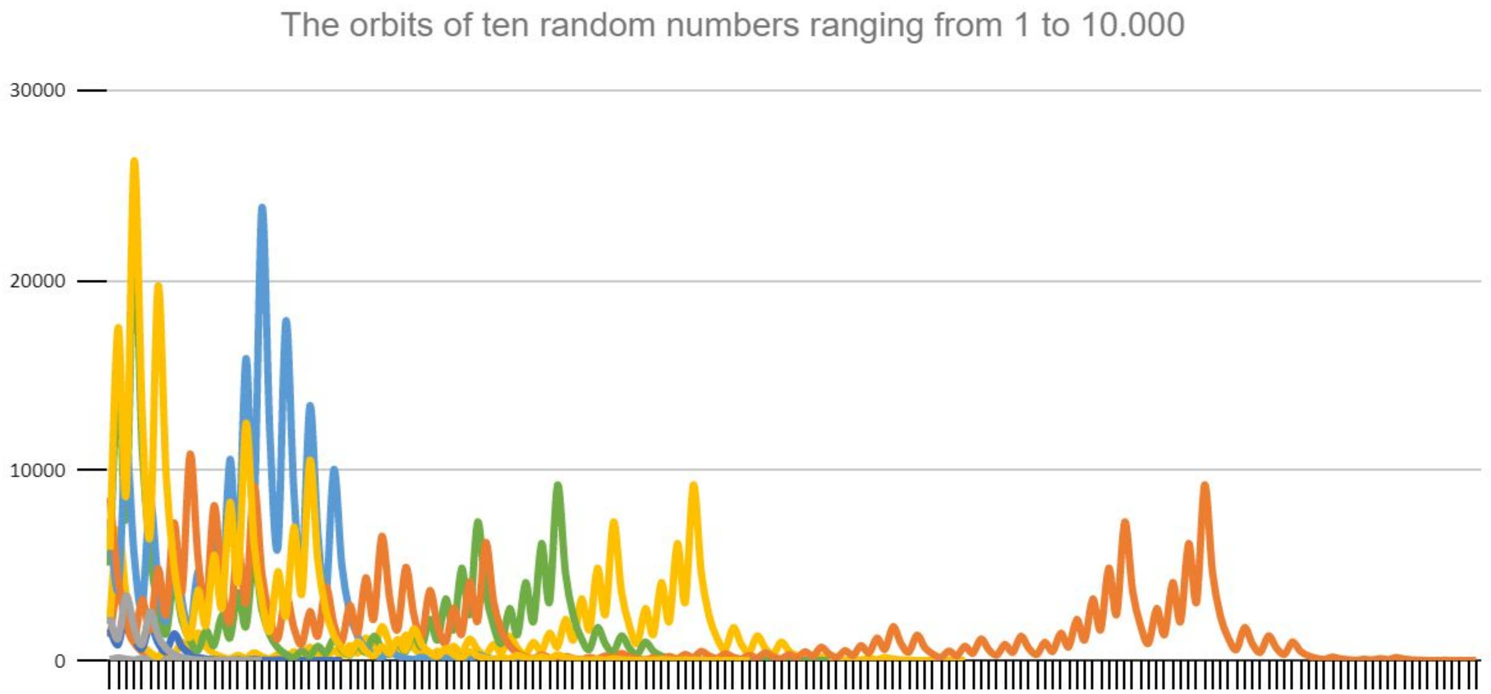
Definition

- Consider iteration of the function *Col*, with n being the initial value, in which the output is fed back into the input

$$n \rightarrow Col(n) \rightarrow Col^2(n) \rightarrow \ldots \rightarrow Col^i(n)$$

$$Col^i(n) = \underbrace{Col(Col\ldots(Col(n)))}_{i\ times}$$

- This process, if repeated multiple times, creates sequences of numbers, known as Collatz Sequences or Collatz Orbits



Example:

11 ⇒ 34 ⇒ 17 ⇒ 52 ⇒ 26 ⇒ 13 ⇒ 40 ⇒ 20 ⇒ 10 ⇒ 5 ⇒ 16 ⇒ 8 ⇒ 4 ⇒ 2 ⇒ 1

SIGNIFICANT RESULTS

Mean Downward trend

- There is a heuristic argument that supports the validity of the conjecture.
- Pick an odd integer n_0 and apply the function Col until you reach another odd number n_1.

½ of the time	$n_1 = (3n_0+1)/2$
¼ of the time	$n_1 = (3n_0+1)/4$
⅛ of the time	$n_1 = (3n_0+1)/8$
…	…

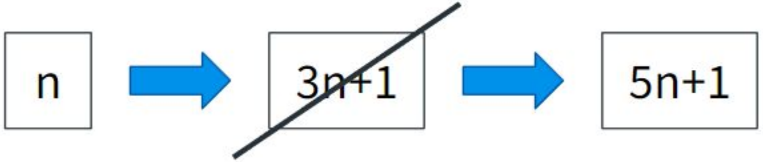
On average	$n_1 \simeq (3/4)\, n_0$

This means that the expected growth in size between two consecutive odd integers in such a trajectory is the multiplicative factor:

$$\left(\frac{3}{2}\right)^{\left(\frac{1}{2}\right)}\left(\frac{3}{4}\right)^{\left(\frac{1}{4}\right)}\left(\frac{3}{8}\right)^{\left(\frac{1}{8}\right)}\left(\frac{3}{16}\right)^{\left(\frac{1}{16}\right)}\left(\frac{3}{32}\right)^{\left(\frac{1}{32}\right)}\cdots = \frac{3^{\sum_{i=0}^{\infty} ar^i}}{2^{\sum_{k=1}^{\infty} kr^k}} = \frac{3^1}{2^2} = \frac{3}{4} < 1$$

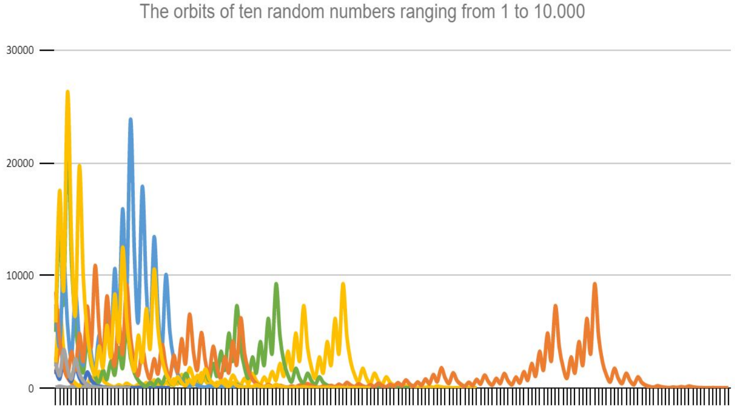
Consequently this heuristic argument suggests that on average the iterates in a trajectory tend to shrink in size

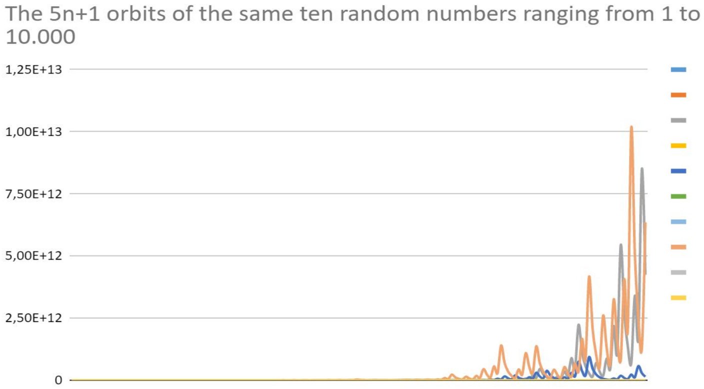
5n+1 sequences diverge



$$\Rightarrow T(n) = \begin{cases} 5n + 1, & if\ n\ is\ odd \\ n/2, & if\ n\ is\ even \end{cases}$$

By repeating the same process with the T(n) function, we get that the expected growth in size between two consecutive odd integers in a 5n+1 trajectory is 5/4, therefore on average the iterates in such a trajectory tend to diverge.





$5 \Rightarrow 16 \Rightarrow 8 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$

$5 \Rightarrow 26 \Rightarrow 13 \Rightarrow 66 \Rightarrow 33 \Rightarrow 166 \Rightarrow 83 \Rightarrow 416 \Rightarrow$ …

Benford’s Law applies to the Collatz Sequences

300 random numbers and their sequences

		Bedford's Law:	
Leading Digit:	**Times found:**	**Actual Percentage:**	**Bedford's Law percentage:**
1	7262	29,831985	30,1
2	4303	17,676539	17,6
3	2793	11,473524	12,5
4	2881	11,835024	9,7
5	1969	8,0885676	7,9
6	1272	5,2253214	6,7
7	1356	5,5703898	5,8
8	1388	5,7018445	5,1
9	1119	4,596804	4,6

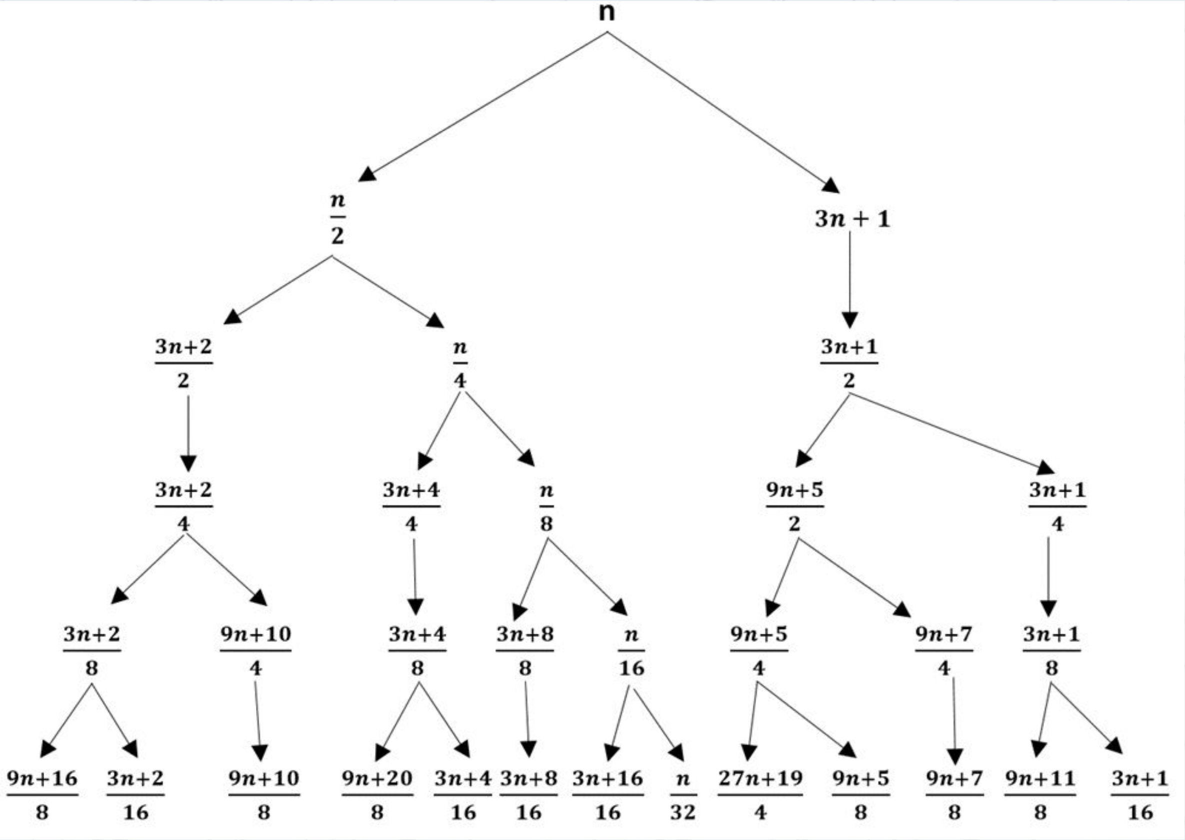
Collatz Circles

- A circle in the collatz conjecture is defined as a closed sequence of numbers, where from one number, after a certain amount of iterations, the sequence returns to that number.

The length **k** of a loop is considered to be the number of steps required for the sequence to return to its initial number.

Hypothetical loop of length 3

The tree diagram



THE COLLATZ FRACTAL

The Collatz Function

We can modify the Collatz function by using an indicator function:

$$\sigma(n) = \begin{cases} 0 \text{ if n is even} \\ 1 \text{ if n is odd} \end{cases} \Rightarrow \sigma(n) = \frac{1 - cos(\pi n)}{2}$$

Recall that $cos(\pi n) = (-1)^n$

Then, we can redefine Col(n) as follows:

$$Col(n) = \begin{cases} 3n+1 \text{ if n is even} \\ n/2 \text{ if n is odd} \end{cases} \Rightarrow [1 - \sigma(n)]\frac{n}{2} + \sigma(n)(3n + 1)$$

$$Col(n) = [1 - \frac{1-cos(\pi n)}{2}]\frac{n}{2} + [\frac{1-cos(\pi n)}{2}](3n + 1)$$

$$Col(n) = \frac{1}{4}[2 + 7n - (2 + 5n)\,cos(\pi n)]$$

The Collatz Function

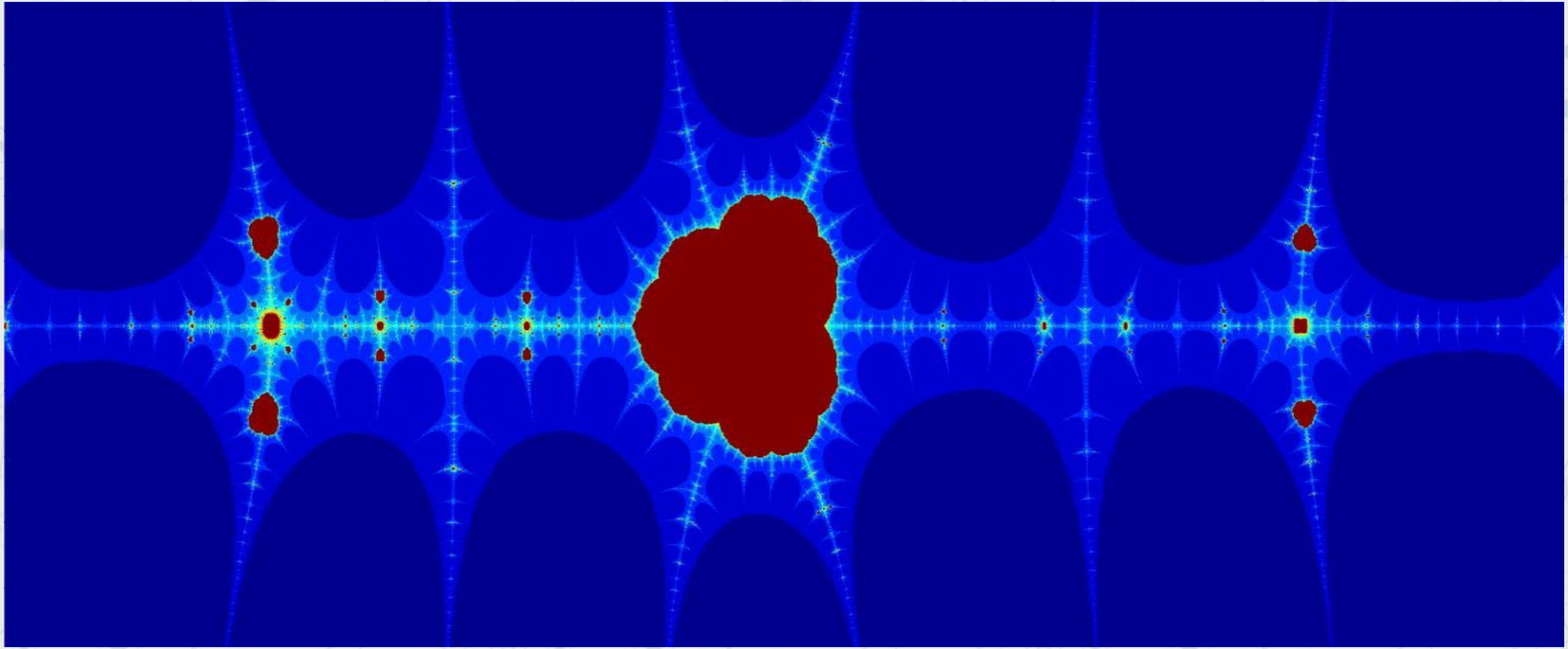
- For x∈C, the Col(x) function is defined as:

$$Col(x) = \frac{1}{4}[2 + 7x - (2 + 5x)cos(\pi x)]$$

- This Function always gives us the next number in a Collatz Sequence for x∈C. For example for x = 7:

$$Col(7) = \frac{1}{4}(2 + 49 - 37(-1)) = \frac{1}{4} \cdot 88 = 22$$

The Collatz Fractal

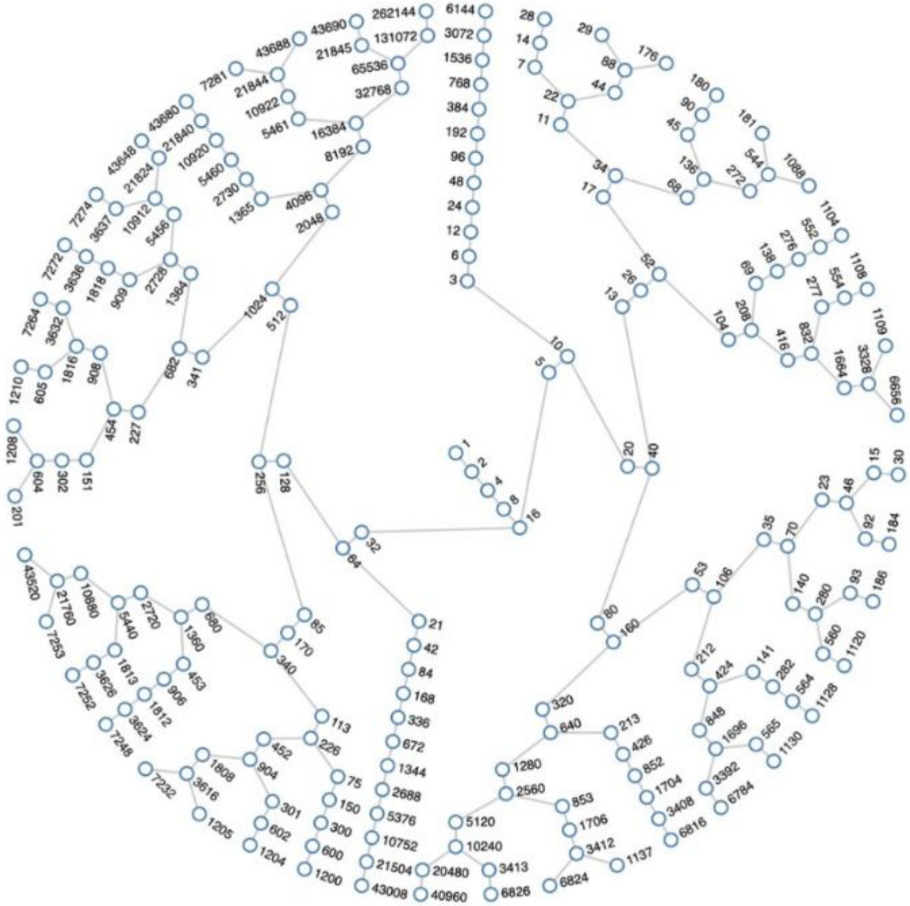


TAO OBSERVATION

Tao Observation

- Terence Tao is a world renowned mathematician. He is a professor of mathematics at the University of California, Los Angeles (UCLA). Tao was awarded the fields medal in 2006.

Terence Tao


6144
3072
1536
768
384
192
96
48
24
12
6
3
262144
131072
43690
21845
65536
32768
43688
21844
10922
5461
16384
8192
7281
43680
21840
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3632
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141
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564
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320
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848
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1704
3392
6784
3408
6816
1280
2560
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3412
1137
5120
10240
3413
6826
6824
20480
40960
43520
21760
10880
5440
2720
7253
1360
680
85
170
340
1813
3626
7252
453
906
1812
3624
7248
113
226
452
904
1808
3616
7232
1205
75
301
150
602
300
1204
600
1200
21
42
84
168
336
672
1344
2688
5376
10752
21504
43008

Tao Observation

- Krasikov and Lagarias → 2003
- For any large number x, there is at least $x^{0.84}$ initial values between 1 and x whose collatz orbits lead to one.
- Tao was able to prove that almost all initial numbers have at least one number in their sequence, smaller than them.

This is as close as one can get to the Collatz Conjecture without actually solving it.

Terence Tao - 2020

Conclusions

1. The Collatz conjecture is a notorious problem in number theory that no one has been able to solve

2. The Collatz conjecture is a test to our understanding of mathematics and number theory

3. There are partial results that support the validity of the conjecture

4. The Collatz fractal could be the breakthrough needed to solve the conjecture

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